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Hochschild Cohomology and Perturbations of Banach Algebras*

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Let A and B be Banach algebras with identity and let $\phi: A \to B$ be a continuous homomorphism. We obtain conditions on the Hochschild cohomology of A under which perturbations of ϕ are similar to ϕ. We also show that if A is a Banach algebra such that $H^2(A, A) = H^3(A, A) = 0$, then perturbations of the multiplication of A give algebras isomorphic to A. We use our techniques to partially answer some problems of Kadison and Kastler on perturbations of operator algebras.

INTRODUCTION

We consider two closely related perturbation problems: If we perturb a homomorphism between two Banach algebras, is the new homomorphism equivalent to the original one? If we perturb the multiplication in a Banach algebra, do we obtain an isomorphic algebra?

These questions were first raised in an operator algebra context by Kadison and Kastler in 1972, [6]. Specifically, they asked:

(1) Are two close representations ϕ and ψ of a C^*-algebra A on a Hilbert space H unitarily equivalent via a unitary close to $1 \in B(H)$?

If so, can we choose the unitary from $(\phi(A) \cup \psi(A))''$, the von Neumann algebra generated by the ranges of ϕ and ψ? By close, we mean the operator norm $\| \phi - \psi \|$ is small.

(2) Are two close von Neumann subalgebras A and B of $B(H)$ unitarily equivalent via a unitary close to $1 \in B(H)$? We say A and B are close if A and B are close in the metric

$$d(A, B) = \sup\{\| a - B_1 \|, \| b - A_1 \| : a \in A_1, b \in B_1\},$$

when A_1 and B_1 denote the unit balls of A and B.

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These questions have been tackled by Christensen [1–3] and Phillips [10]. They have obtained positive results under various restrictions on the algebras; in particular, Christensen has answered the second problem under a variety of hypotheses on A and B. As yet no counterexample to either problem is known.

We first tackle these problems in Banach algebras, and then use the same techniques to obtain partial answers to the Kadison–Kastler problems. We show that if certain Hochschild cohomology groups vanish, then the perturbation problems in Banach algebras have a positive answer. In the case of the first problem, under the same hypotheses the Banach algebra problem and the C^*-algebra problem have solutions. We do not know if every C^*-algebra satisfies these hypotheses; but we do know that amenable C^*-algebras do. The second Kadison–Kastler problem is related to the second Banach algebra problem in a less obvious way. We first show that a $*$-multiplication on a Banach $*$-algebra A, close to the original one, gives a $*$-isomorphic algebra provided the Hochschild groups $H^2(A, A)$ and $H^3(A, A)$ vanish. We then use this result to show that if, also, A is a von Neumann subalgebra of $B(H)$ with the extension property, then a close von Neumann subalgebra B must be $*$-isomorphic to A. We can then use our answer to the first problem to obtain a partial answer to the Kadison–Kastler question.

In Section 1 we review the Hochschild cohomology theory of a Banach algebra, and introduce other concepts we shall need later. Then we prove in Section 2 a new version of the implicit function theorem which will give the connection between Hochschild cohomology and the perturbation problems. In Sections 3 and 4 we apply the implicit function theorem to the first and second problems, respectively.

As this paper was being prepared, we learned from Barry Johnson that he had proved most of our results independently at about the same time. Since our approach is somewhat different, his results will appear in a separate paper. It will contain in particular the results of our Section 4 on perturbations of algebras. He also has a proof of our Theorem 2 in the case where the second algebra is $B(X)$ for some Banach space X.

1. Preliminaries

Let A be a Banach algebra, and let M be a two-sided A-module. Suppose further that M is a Banach A-module—that is, M is a Banach space and the module operations are continuous. Let $L^n(A, M)$ for $n \geqslant 1$ denote the Banach space of all continuous n-linear maps of A^n into M, and let $L^0(A, M) = M$. We define $\delta^n: L^{n-1}(A, M) \to L^n(A, M)$ for $n \geqslant 1$ by

$$(\delta^n T)(a_1, \ldots, a_n) = a_1 T(a_2, \ldots, a_n) - T(a_1 a_2, a_3, \ldots, a_n) + T(a_1, a_2 a_3, \ldots, a_n) - \cdots + (-1)^{n-1} T(a_1, \ldots, a_{n-1} a_n) + (-1)^n T(a_1, \ldots, a_{n-1}) a_n,$$

for $T \in L^{n-1}(A, M)$ and $(a_1, \ldots, a_n) \in A^n$. Then $\delta^{n+1}\delta^n = 0$ for all $n \geqslant 1$, so that

$$L^0(A, M) \xrightarrow{\delta^1} L^1(A, M) \xrightarrow{\delta^2} L^2(A, M) \xrightarrow{\delta^3} \cdots$$

is a complex of Banach spaces, which we call the Hochschild complex for A with coefficients in M. Similarly, we call the groups

$$H^n(A, M) = \frac{\ker \delta^{n+1}}{\text{range } \delta^n} \quad \text{for} \quad n \geqslant 1,$$

the Hochschild cohomology groups for A with coefficients in M. If M is a Banach A-module, then its dual Banach space M^* is also a Banach A-module under the operations

$$af(m) = f(ma) \qquad \text{and} \qquad fa(m) = f(am) \qquad \text{for} \quad a \in A, f \in M^*, m \in M.$$

If $H^n(A, M^*) = 0$ for every Banach A-module M and for every n, we call A an amenable Banach algebra. For further details, we refer to Johnson's monograph [4].

Let A be a von Neumann algebra acting on a Hilbert space H. We say A has the extension property of there is a linear projection of norm 1 of $B(H)$ onto A. In particular, the results of [11, Sect. 4.4] imply that type I and hyperfinite algebras have the extension property. In general, our terminology in matters concerning operator algebras is that of [11].

2. An Implicit Function Theorem

Let X and Y be real Banach spaces and $U \subset X$ a domain. In what follows we deal with maps $f\colon U \to Y$ which are C^2 in the Fréchet sense. Thus, the derivative of f is a continuous map

$$u \to f'(u)\colon U \to L(X, Y).$$

The second derivative is a continuous map $u \to f''(u)\colon U \to L^2(X, Y)$, the space of continuous bilinear maps from X to Y. It follows from Taylor's formula that

$$\| f(u + x) - f(u) - f'(u)x \| \leqslant \tfrac{1}{2}K \| x \|^2$$

whenever the line segment $\{u + tx\colon t \in [0, 1]\}$ belongs to U and K is a bound for $\| f'' \|$ on this line segment. For the details, see [9].

If $g\colon X \to Y$ is a linear map with closed range, then the induced map $\tilde{g}$: $X/\ker g \to \operatorname{im} g$ has a bounded inverse. We call the norm of this inverse the inversion constant of g.

THEOREM 1. *Let X, Y, Z be Banach spaces and $U \subset X$, $V \subset Y$ domains. Let $f: U \to V$ and $k: V \to Z$ be C^2 maps and $u_0 \in U$, $v_0 \in V$ points with $f(u_0) = v_0$. Suppose further that*

(a) *$k \circ f$ is constant;*

(b) $\operatorname{im} f'(u_0) = \ker k'(f(u_0))$;

(c) *$k'(f(u))$ has closed range for $u \in U$ and inversion constant uniformly bounded over U;*

then there exists $\delta > 0$ and $C > 0$ such that for each $v \in V$ with $\| v_0 - v \| < \delta$ and $k(v) = k(v_0)$ there is a $u \in U$ with $\| u - u_0 \| \leqslant C \| v - v_0 \|$ and $f(u) = v$.

Proof. Differentiating equation (a) yields that $k'(f(u)) \circ f'(u) = 0$ for $u \in U$. That is, $\operatorname{im} f'(u) \subset \ker k'(f(u))$. By (b) the containment is an equality at u_0. This, together with (c) and the proof of Lemma 2.1 of [12] implies that $\operatorname{im} f'(u) = \ker k'(f(u))$ for all u in a neighborhood of u_0 and, furthermore, that the inversion constant for $f'(u)$ is bounded on this neighborhood. Hence, by shrinking U if necessary, we may assume there is a constant M such that for $u \in U$:

$$\begin{gathered}\text{for each } y \in \ker k'(f(u)) \text{ there exists } x \in X \\ \text{with } f'(u)x = y \text{ and } \| x \| \leqslant M \| y \|;\end{gathered} \tag{1}$$

$$\begin{gathered}\text{for each } z \in \operatorname{im} k'(f(u)) \text{ there exists } y \in Y \\ \text{with } k'(f(u))y = z \text{ and } \| y \| \leqslant M \| z \|.\end{gathered} \tag{2}$$

Since f and k are C^2, by shrinking U and V if necessary, we may assume that U and V are convex neighborhoods of u_0 and v_0 on which f'' and k'' are bounded. It follows that there is a constant K such that if u, $u + x \in U$ and v, $v + y \in V$ then

$$\| f(u + x) - f(u) - f'(u)x \| \leqslant K \| x \|^2; \tag{3}$$

$$\| k(v + y) - k(v) - k'(v) y \| \leqslant K \| y \|^2. \tag{4}$$

Now let $v \in V$ satisfy $k(v) = k(v_0)$ and $\| v - v_0 \| < \delta$ where δ will be specified later in the proof. We consider u_0 to be an initial approximate solution to the equation $f(u) = v$ and proceed as in Newton's method. Since $f(u_0) = v_0$, $v - v_0$ is our initial error. By (2) there exists $y_1 \in Y$ with

$$\begin{gathered} k'(v_0) y_1 = k'(v_0)(v - v_0) \\ \| y_1 \| \leqslant M \| k'(v_0)(v - v_0)\|. \end{gathered}$$

But since $k(v_0) = k(v)$ we have

$$\| k'(v_0)(v - v_0)\| = \| k(v) - k(v_0) - k'(v_0)(v - v_0)\| \leqslant K \| v - v_0 \|^2$$

by (4). Thus,

$$\| y_1 \| \leqslant MK \| v - v_0 \|^2. \tag{5}$$

Since $k'(v_0)(v - v_0 - y_1) = 0$, it follows from (1) that there exists $x_1 \in X$ with

$$f'(u_0)\, x_1 = v - v_0 - y_1$$
$$\| x_1 \| \leqslant M \| v - v_0 - y_1 \|.$$

If $\delta < 1$ then

$$\begin{aligned} \| x_1 \| &\leqslant M \| v - v_0 \| + M \| y_1 \| \\ &\leqslant (M + M^2K) \| v - v_0 \|. \end{aligned} \tag{6}$$

Provided it lies in U, we may choose $u_1 = u_0 + x_1$ as our next approximate solution and set $v_1 = f(u_1)$. Our error is then from (3), (5), and (6):

$$\begin{aligned} \| v - v_1 \| &= \| v - f(u_0 + x_1) \| \\ &\leqslant \| v - v_0 - f'(u_0)\, x_1 \| + \| f(u_0 + x_1) - f(u_0) - f'(u_0)\, x_1 \| \\ &\leqslant \| y_1 \| + K \| x_1 \|^2 \\ &\leqslant (MK + K(M + M^2K)^2) \| v - v_0 \|^2 \\ &\leqslant r \| v - v_0 \|, \end{aligned} \tag{7}$$

where $r = (MK + K(M + M^2K)^2)\delta$.

If $r < 1$ we may iterate the procedure, replacing u_0 and v_0 by u_1 and v_1 in the above argument and obtaining u_2 and v_2. Proceeding in this way, we generate sequences $\{u_n\} \subset U$ and $\{v_n\} \subset V$ such that

$$\| u_{n+1} - u_n \| \leqslant (M + M^2K) \| v - v_n \| \leqslant (M + M^2K)\, r^n \| v - v_0 \|$$
$$\| v - v_n \| \leqslant r^n \| v - v_0 \|,$$

provided we can be assured at each stage that $u_n \in U$. Since

$$u_n = u_0 + \sum_{i=1}^{n} (u_i - u_{i-1}),$$

we have

$$\| u_n - u_0 \| \leqslant C \| v - v_0 \|,$$

where

$$C = (1 - r)^{-1}(M + M^2K).$$

Thus, if δ is chosen small enough that $r < 1$ and $C\delta$ is less than the distance from u_0 to the complement of U, we will have $u_n \in U$ at each stage, u_n converges to $u \in U, f(u) = \lim v_n = v$, and $\| u - u_0 \| \leqslant C \| v - v_0 \|$.

3. Perturbations of Representations

Let A and B be Banach algebras with identity and let $\phi: A \to B$ be an identity preserving continuous algebra homomorphism. Then the operations $(a, b) \to \phi(a)b$ and $(b, a) \to b\phi(a)$ give B the structure of a 2-sided A-module which we denote by B_ϕ.

THEOREM 2. (a) *If* $H^1(A, B_\phi) = H^2(A, B_\phi) = 0$ *then there are constants* $\delta > 0$, $C > 0$ *such that if* $\psi: A \to B$ *is a continuous homomorphism with* $\|\phi - \psi\| < \delta$ *then* $\psi(a) = b^{-1}\phi(a)b$ *for some* $b \in B$ *with* $\|1 - b\| \leqslant C\|\phi - \psi\|$. (b) *If, in addition,* A *and* B *are *-algebras and* ϕ *and* ψ *are *-homomorphisms, then* b *can be chosen to be unitary.*

Proof. Define $f: B^{-1} \to L(A, B)$ and $k: L(A, B) \to L^2(A, B)$ by

$$f(b)a = b^{-1}\phi(a)b,$$

$$k(\psi)(a_1, a_2) = \psi(a_1)\psi(a_2) - \psi(a_1a_2).$$

Then f and k are C^2 maps with $k \circ f = 0$ and $k^{-1}(0)$ is the space of homomorphisms from A to B while $\operatorname{im} f$ consists of those homomorphisms similar to ϕ. The derivatives $f'(1)$ and $k'(f(1)) = k'(\phi)$ are the first and second coboundary maps in the Hochschild complex for A with coefficients in B_ϕ. Since $H^1(A, B_\phi) = 0$ we have $\operatorname{im} f'(1) = \ker k'(f(1))$. Further, for $b \in B^{-1}$ the derivative $k'(f(b))$ is the second Hochschild coboundary operator for A with coefficients in $B_{f(b)}$. Since B_ϕ and $B_{f(b)}$ are isomorphic modules via the map $b_1 \to b^{-1}b_1b: B \to B$ and since $H^2(A, B_\phi) = 0$, we have that each $k'(f(b))$ has closed range. Also, the inversion constant for $k'(f(b))$ can be computed in terms of that for $k'(\phi)$ using this isomorphism, and we conclude that for b in a neighborhood of 1 these constants are bounded. We may now apply the implicit function theorem and (a) follows at once.

For part (b) we let

$$X = \{b \in B: b^* = -b\},$$
$$Y = \{\psi \in L(A, B): \psi(a^*) = \psi(a)^*\},$$
$$Z = \{\alpha \in L^2(A, B): \alpha(a_1^*, a_2^*) = \alpha(a_2, a_1)^*\}.$$

We define $f: X \to Y$ and $k: Y \to Z$ by

$$f(b)a = e^{-b}\phi(a)e^b,$$

$$k(\psi)(a_1, a_2) = \psi(a_1)\psi(a_2) - \psi(a_1a_2).$$

Then $f'(0)$ and $k'(f(0))$ are the first and second Hochschild coboundary maps restricted to the real linear subspaces X and Y of B and $L(A, B)$. It is easy to see this restriction preserves exactness and, hence, that Theorem 1 still applies. Thus (b) follows from the observation that e^b is unitary if $b^* = -b$.

COROLLARY. *Let* A *be an amenable* C^**-algebra and* ϕ *be a *-representation of* A *on a Hilbert space* H. *Then there are constants* $\delta > 0$, $C > 0$ *such that if* ψ *is a *-representation of* A *on* H *with* $\|\phi - \psi\| < \delta$ *then* $\psi(a) = u^*\phi(a)u$ *for some unitary* $u \in (\phi(A) \cup \psi(A))''$ *with* $\|1 - u\| \leqslant C\|\phi - \psi\|$.

Proof. By the theorem, if A has an identity, it is enough to show that $H^1(A, B_\phi) = H^2(A, B_\phi) = 0$ where $B = (\phi(A) \cup \psi(A))''$. But B is a von Neumann algebra and so is a dual space. It is easy to check that it is in fact a dual module so that $H^n(A, B_\phi) = 0$ for all n. We can drop the condition that A have an identity by observing that A is amenable if and only if $A \otimes \mathbb{C}1$ is amenable.

Christensen [2, 3] has obtained this result under various hypotheses on A; in particular, if A is strongly amenable. It is not yet known if there are C^*-algebras which are amenable but not strongly amenable. Bunce has recently proved that there are nonamenable C^*-algebras, and so our result is not a complete answer to Kadison and Kastler's question.

4. Perturbations of Banach Algebras

In what follows, A is a Banach algebra which may or may not have an identity.

THEOREM 3. (a) *If $H^2(A, A) = H^3(A, A) = 0$ then there are constants $\delta > 0$, $C > 0$ such that if m is any associative multiplication on A with $\| m(a, b) - ab \| < \delta$ for $a, b \in A$, $\| a \| \leqslant 1$, $\| b \| \leqslant 1$, then there exists $\phi \in L(A)^{-1}$ with $\| \phi - I \| \leqslant C \sup\{\| m(a, b) - ab \|: \| a \| \leqslant 1, \| b \| \leqslant 1\}$ and $\phi(m(a, b)) = \phi(a)\phi(b)$.*

(b) *If, in addition, A is a Banach *-algebra and m defines a *-multiplication on A then ϕ can be chosen so that $\phi(a^*) = \phi(a)^*$.*

Proof. Define $f: L(A)^{-1} \to L^2(A, A)$ and $k: L^2(A, A) \to L^3(A, A)$ by

$$f(\phi)(a, b) = \phi^{-1}(\phi(a)\phi(b)),$$
$$k(\alpha)(a, b, c) = \alpha(a, \alpha(b, c)) - \alpha(\alpha(a, b), c).$$

Then $k \circ f = 0$, $k^{-1}(0)$ is the set of associative multiplications on A and $\operatorname{im} f$ is the set of multiplications isomorphic to the given one. The maps f and k are C^2 and the complex

$$L(A) \xrightarrow{f'(1)} L^2(A, A) \xrightarrow{k'(f(1))} L^3(A, A)$$

comprises the second and third stages of the Hochschild complex for A with coefficients in A. Further, for $\phi \in L(A)^{-1}$, $k'(f(\phi))$ is the third Hochschild coboundary operator for the algebra A with multiplication determined by $f(\phi)$. Since ϕ gives an isomorphism between A and this second algebra we conclude that $k'(f(\phi))$ has closed range for each ϕ and that the inversion constants are bounded for ϕ near 1. Hence, as in the proof of Theorem 2, we have the hypotheses of our implicit function theorem satisfied. Part (a) follows.

For part (b), we proceed as in Theorem 2, replacing $L(A)$, $L^2(A, A)$, and $L^3(A, A)$ by the real linear subspaces

$$X = \{\phi \in L(A): \phi(a)^* = \phi(a^*)\},$$
$$Y = \{\alpha \in L^2(A, A): \alpha(b^*, a^*) = \alpha(a, b)^*\},$$
$$Z = \{\rho \in L^3(A, A): \rho(c^*, b^*, a^*) = -\rho(a, b, c)^*\}.$$

COROLLARY. *Let A be a closed *-subalgebra of $B(H)$ such that $H^2(A, A) = H^3(A, A) = 0$. Then there are constants $\delta > 0$, $C > 0$ such that if B is another *-subalgebra of $B(H)$ and λ: $A \to B$ a linear isomorphism satisfying $\|\lambda - i\| < \delta$, where i: $A \to B(H)$ is the inclusion, then there is a *-isomorphism ϕ of A onto B with $\|\phi - \lambda\| \leqslant C\|\lambda - i\|$.*

Proof. We may assume $\lambda(a^*) = \lambda(a)^*$ since otherwise we may replace λ by $\tilde{\lambda}$ where $\tilde{\lambda}(a) = \frac{1}{2}(\lambda(a) + \lambda(a^*)^*)$. Then the multiplication m on A given by $m(a_1, a_2) = \lambda^{-1}(\lambda(a_1)\lambda(a_2))$ defines a *-multiplication on A close to the original one. Hence, Theorem 3 gives us a *-map $\phi \in L(A)^{-1}$ such that

$$\lambda^{-1}(\lambda(a_1)\lambda(a_2)) = \phi^{-1}(\phi(a_1)\phi(a_2)).$$

It follows that $\lambda \circ \phi^{-1}$ is a *-isomorphism of A onto B. The estimate of Theorem 3 gives us a bound on $\|\phi - \lambda\|$.

COROLLARY. *Let A be a von Neumann subalgebra of $B(H)$ such that $H^2(A, A) = H^3(A, A) = 0$ and suppose that A has the extension property. Then there are constants $\delta > 0$, $C > 0$ such that if B is another von Neumann subalgebra of $B(H)$ and $d(A, B) < \delta$ then there exists a *-isomorphism ϕ: $A \to B$ with $\|\phi - i\| \leqslant Cd(A, B)$, where i: $A \to B(H)$ is the injection.*

Proof. Since A has the extension property, there is a linear projection P of $B(H)$ onto A of norm one. Then if B is a von Neumann algebra close to A, $P|_B$ will be a linear isomorphism of B onto A which is close to the identity. The inverse of $P|_B$ will do for the λ of the previous corollary and the result follows.

Remarks. In the previous two corollaries one can conclude that the isomorphism ϕ is actually implemented by a unitary u in $B(H)$ (that is $\phi(a) = uau^*$) if $H^1(A, B(H)) = H^2(A, B(H)) = 0$. This follows directly from Theorem 2.

In the second corollary the hypothesis that A have the extension property is only used to ensure that there is a linear isomorphism of A onto B. This raises the question: Under what circumstances are two close subspaces of a Banach space necessarily isomorphic?

In [7, 8], Kadison and Ringrose have shown that $H^n(A, A) = 0$ for all n when A is either a type I or hyperfinite von Neumann algebra; hence all the hypotheses of the last corollary are satisfied if A is type I or hyperfinite. In the case when A is type I, the result was proved independently by Christensen [1] and Phillips [10]. The other results of Christensen [2, 3] are not strictly comparable with ours. He imposes conditions on both subalgebras A and B; in each case he considers we can obtain the linear isomorphism of A onto B we require, but we do not know if the necessary cohomology groups vanish.

Note added in proof. Erik Christensen and John Phillips have pointed out to us that the condition $H^2(A, A) = H^3(A, A) = 0$ in the second corollary of section 4 is redundant. For suppose P is the projection of norm one of $B(H)$ onto A; then by a result of Tomiyama [*Proc. Japan Acad.* **33** (1957), 608–612] $P(xy) = xP(y)$ and $P(yx) = P(y)x$ for all $x \in A$ and $y \in B(H)$. It now follows from [J. R. Ringrose, Cohomology of operator algebras, Springer-Verlag Lecture notes in mathematics no. 249, p. 429] that $H^n(A, A) = 0$ for all n.

Allan Sinclair has pointed out to us that condition (c) in Theorem 1 may be replaced by the weaker condition (*c'*) $k'(f(u_0))$ *has closed range*. The theorem follows as before, except that in place of Lemma 2.1 of [12] we use the following Lemma, which is a slight strengthening of Lemma 6.1 of [5]. The proof of [5, Lemma 6.1] also gives this result.

Lemma. Let $X_i (i = 1, 2, 3)$ be Banach spaces and let $S_i, T_i \in L(X_i, X_{i+1}) (i = 1, 2)$. Suppose that $1mS_1 = \ker S_2$, $1mS_2$ is closed, S_i has inversion constant $K_i (i = 1, 2)$ and that $T_2 T_1 = 0$. Then if

$$k = K_1 \| S_1 - T_1 \| + K_2 \| S_2 - T_2 \| + K_1 K_2 \| S_1 - T_1 \| \| S_2 - T_2 \| < 1$$

we have $1mT_1 = \ker T_2$, $1mT_2$ is closed and that the inversion constants for T_1 and T_2 are bounded by $K_1(1 + K_2 \| S_2 - T_2 \|)/(1 - k)$ and $K_2(1 + K_1 \| S_1 - T_1 \|)/(1 - k)$ respectively.

References

1. E. Christensen, Perturbations of type I von Neumann algebras, *J. London Math. Soc.* **9** (1975), 395–405.
2. E. Christensen, Perturbations of operator algebras, University of Oslo Preprint Series, No. 14, 1974.
3. E. Christensen, Perturbations of operator algebras, II, University of Copenhagen Preprint Series, No. 6, 1975.
4. B. E. Johnson, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.* **127**, 1972.
5. B. E. Johnson, Perturbations of Banach algebras, to appear.
6. R. V. Kadison and D. Kastler, Perturbations of von Neumann algebras I, Stability of type, *Amer. J. Math.* **94**, No. 1 (1972), 38–54.
7. R. V. Kadison and J. R. Ringrose, Cohomology of operator algebras, I: Type I von Neumann algebras, *Acta Math.* **126** (1971), 227–243.
8. R. V. Kadison and J. R. Ringrose, Cohomology of operator algebras, II, Extended cobounding and the hyperfinite case, *Ark. Mat.* **9** (1971), 55–63.
9. S. Lang, "Analysis II," Addison–Wesley, Reading, Mass., 1969.
10. J. Phillips, Perturbations of type I von Neumann algebras, *Pacific J. Math.* **52** (1974), 505–511.
11. S. Sakai, "C^*-Algebras and W^*-Algebras," Ergeb. der Math., Vol. 60, Springer-Verlag, Berlin, 1971.
12. J. L. Taylor, A joint spectrum for several commuting operators, *J. Functional Anal.* **6** (1970), 172–191.